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A new proof of the affine isoperimetric inequality for Orlicz mean zonoids

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Abstract

The Orlicz mean zonoid is defined by Guo, Leng and Du, and establish an affine isoperimetric inequality for it. Using shadow systems, we provide a new proof of the affine isoperimetric inequality for the Orlicz mean zonoids.

Keywords: The shadow system; zonoid; Orlicz mean zonoid; affine isoperimetric inequality; Steiner symmetrization.

1 Introduction

Motivated by recent progress in the asymmetric L_p-Brunn-Minkowski theory (see e.g. [7, 8, 9, 14, 17, 18, 22, 23]), Lutwak, Yang, and Zhang introduced the Orlicz Brunn-Minkowski theory in two articles [12, 13]. Since this seminal work, this new theory has evolved rapidly (see e.g. [4, 6, 10, 11, 15, 16, 24]. Not long ago, Guo, Leng and Du in [5] introduced an Orlicz mean zonoid, and establish an affine isoperimetric inequality for the Orlicz mean zonoid. In this paper, inspired by the work of Campi, Gronchi [1, 2, 3] and Li et. al.[16], we will give an alternative proof of the affine isoperimetric inequality for the Orlicz mean zonoid.

In order to state the results regarding to the Orlicz mean zonoid bodies, several notation are needed. For a compact convex subset K in $\mathbb{R}^n$, let $h(K, \cdot) = h_K(\cdot) : \mathbb{R}^n \to \mathbb{R}$ denotes the support function of K; that is,

$$h_K(x) = \max\{\langle x, y\rangle : y \in K\},\ x \in \mathbb{R}^n,$$

where $\langle x, y\rangle$ denotes the inner product of x and y in $\mathbb{R}^n$. For $c > 0$, the support function of the convex body $cK = \{cx : x \in K\}$ is $h_{cK} = ch_K$. For $T \in \mathrm{GL}(n)$, it follows that

$$h_{TK}(x) = h_K(T^t x). \tag{1.1}$$

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The Hausdorff distance $\delta(K, L)$ between the convex bodies K and L is

$$\delta(K,L) = ||h_K - h_L||_\infty = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

We write $\mathcal{K}^n$ for the set of convex bodies in $\mathbb{R}^n$, and write $\mathcal{K}_o^n$ for the set of convex bodies that contain the origin in their interiors.

The radial function $\rho(K,\cdot) = \rho_K(\cdot) : \mathbb{R}^n \backslash \{o\} \to [0,\infty)$, of a compact star-shaped about the origin $K \subset \mathbb{R}^n$, is defined by

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}.$$

If ρ_K is positive and continuous, then K is called a star body about the origin.

Let $\phi : \mathbb{R} \to [0,\infty)$ be an even convex function such that $\phi(0) = 0$. This means that ϕ must be decreasing on $(-\infty, 0]$ and increasing on $[0,\infty)$. The set of such ϕ is denoted by $\mathcal{C}$. Let $K \subset \mathbb{R}^n$ be a star body about the origin with volume $V(K)$ and $\phi \in \mathcal{C}$, the Orlicz centroid body $\Gamma_\phi K$ of K is a convex body whose support function at $x \in \mathbb{R}^n$ is defined by (see [12])

$$h_{\Gamma_\phi K}(x) = \inf\left\{\lambda > 0 : \frac{1}{V(K)} \int_K \phi\left(\frac{\langle x, y\rangle}{\lambda}\right) \mathrm{d}y \leq 1\right\},$$

where the integration is with respect to Lebesgue measure in $\mathbb{R}^n$.

The Orlicz centroid body is the natural generalization of the L_p-centroid body $\Gamma_p K$ and the centroid body ΓK. For $p \geq 1$, let $\phi(t) = \phi_p(t) = |t|^p$, then

$$\Gamma_{\phi_p} K = \Gamma_p K.$$

Lutwak et al. in [12] obtain the following Orlicz Busemann-Petty centroid inequality. If $\phi \in \mathcal{C}$ and $K \in \mathcal{K}_o^n$, then the volume ratio

$$V(\Gamma_\phi K)/V(K)$$

is minimized if and only if K is an ellipsoid centered at the origin. Using shadow systems, Li and Leng [16] provide a new proof of the Orlicz Busemann-Petty centroid inequality.

Recently, Guo and Leng et al.[5] introduced an Orlicz mean zonoid operator Z_ϕ which is on Orlicz generalization of the mean zonoid operator of Zhang [25]. Let $K, L \in \mathcal{K}^n$ and $\phi \in \mathcal{C}$. The Orlicz mean zonoid body $Z_\phi(K,L)$ of K and L as the convex body whose support function at $x \in \mathbb{R}^n$ is defined by

$$h_{Z_\phi(K,L)}(x) = \inf\left\{\lambda > 0 : \frac{1}{V(K)V(L)} \int_K \int_L \phi\left(\frac{\langle x, (y-z)\rangle}{\lambda}\right) \mathrm{d}y\mathrm{d}z \leq 1\right\}, \tag{1.2}$$

where the integration is with respect to Lebesgue measure in $\mathbb{R}^n$.

Taking $K = L$ in (1.2), then

$$h_{Z_\phi K}(x) := h_{Z_\phi(K,K)}(x) = \inf\left\{\lambda > 0 : \frac{1}{V(K)^2} \int_K \int_K \phi\left(\frac{\langle x, (y-z)\rangle}{\lambda}\right) \mathrm{d}y\mathrm{d}z \leq 1\right\}.$$

Guo and Leng et al.[5] established the following general affine isoperimetric inequality for for Orlicz mean zonoids: If $K, L \in \mathcal{K}_o^n$ and $\phi \in \mathcal{C}$, then

$$V(Z_\phi(K,L)) \geq V(Z_\phi(B_K, B_L)), \tag{1.3}$$

with equality if K and L are dilated ellipsoids having the same midpoints, where B_K denote the n-ball with the same volume as K centered at the origin.

Motivated by the idea of Campi and Gronchi [1], Li and Leng [16], our job is to provide a new proof of the affine isoperimetric inequality for the Orlicz mean zonoids. The critical part of the proof in [5] is that the volume of the Orlicz mean zonoid body is not increased after a Steiner symmetrization. It is well known that every convex body can be transformed into a ball by a sequence of suitable Steiner symmetrizations. In this paper, we also follow this principle. The technique we will use is that of shadow systems developed by Rogers [20] and Shephard [21].

2 Shadow system of convex body

A shadow system (or a linear parameter system) along the direction v is a family of convex sets $K_t \subset \mathbb{R}^n$ that can be defined by (see [20, 21])

$$K_t = \mathrm{conv}\{z + \alpha(z)tv : \ z \in A \subset \mathbb{R}^n\}, \tag{2.1}$$

where A is an arbitrary bounded closed set of points, α is a real bounded function on A, and the parameter t runs in an interval of the real axis.

For a convex body K in $\mathbb{R}^n$, a parallel chord movement along the direction v is a shadow system defined by

$$K_t = \{z + \beta(z|v^\perp)tv : \ z \in K, t \in [0,1]\}, \tag{2.2}$$

where β is a continuous real function on the projection $K|v^\perp$ of K onto $v^\perp = \{z \in \mathbb{R}^n : \langle v, z\rangle = 0\}$. In other words, to each chord of $K = K_0$ parallel to v we assign a speed vector $\beta(x)v$, where x is the projection of the chord onto $v^\perp$, then let the chords move for a time t and denote by K_t their union. Such a union has to be convex, this is the only restriction we have on defining the speed function β.

Let K be a convex body in $\mathbb{R}^n$. For $u \in S^{n-1}$, the Steiner symmetrization S_uK with respect to the hyperplane $u^\perp$ is the set generated by translating all chords of K that are parallel to u so that their midpoints are on the hyperplane $u^\perp$.

Another instance is the movement related to Steiner symmetrization. For a direction v and let

$$K = \{x + yv \in \mathbb{R}^n : \ x \in K|v^\perp, y \in \mathbb{R}, \ f_y(x) \le y \le g_y(x)\}, \tag{2.3}$$

here f_y and $-g_y$ are convex functions on $K|v^\perp$. If one takes $\beta(x) = -(f_y(x) + g_y(x))$ in (2.2) and $t \in [0,1]$ such that $K_0 = K$ and $K_1 = K^v$, where K^v is the reflection of K in the hyperplane $v^\perp$, and $K_{1/2} = S_vK$ is the Steiner symmetrization of K with respect to $v^\perp$.

The following lemma is due to Shephard [21].

Lemma 2.1. *Every mixed volume involving n shadow systems along the same direction is a convex function of the parameter. In particular, the volume $V(K_t)$ and all quermassintegrals $W_i(K_t), i = 1, 2, \cdots, n$, of a shadow system are convex functions of t.*

Lemma 2.2. (see [5]) *Suppose $K, L \in \mathcal{K}^n$ and $\phi \in \mathcal{C}$. Then for $x_0 \in \mathbb{R}^n \backslash \{o\}$,*

$$h_{Z_\phi(K,L)}(x_0) = \lambda_0 \quad \Leftrightarrow \quad \frac{1}{V(K)V(L)} \int_K \int_L \phi\Big(\frac{\langle x_0, (y - z)\rangle}{\lambda_0}\Big) \mathrm{d}y \mathrm{d}z = 1. \tag{2.4}$$

The operator Z_ϕ intertwines with elements of GL(n):

Lemma 2.3. *Let $\phi \in \mathcal{C}$. For two convex bodies $K, L \in \mathcal{K}^n$ and a linear transformation $T \in \mathrm{GL}(n)$,*

$$Z_\phi(TK, TL) = T(Z_\phi(K, L)).$$

Proof. Taking $y_1 = T^{-1}y$ and $z_1 = T^{-1}z$, by $V(TK) = |T|V(K), \mathrm{d}y = |T|\mathrm{d}y_1$, et. al., and (1.1), we have

$$\begin{aligned}
h_{Z_\phi(TK,TL)}(x) &= \inf\left\{\lambda > 0 : \frac{1}{V(TK)V(TL)}\int_{TK}\int_{TL}\phi\Big(\frac{\langle x, (y-z)\rangle}{\lambda}\Big)\mathrm{d}y\mathrm{d}z \le 1\right\}\\
&= \inf\left\{\lambda > 0 : \frac{1}{V(K)V(L)}\int_{K}\int_{L}\phi\Big(\frac{\langle x, (Ty_1-Tz_1)\rangle}{\lambda}\Big)\mathrm{d}y_1\mathrm{d}z_1 \le 1\right\}\\
&= \inf\left\{\lambda > 0 : \frac{1}{V(K)V(L)}\int_{K}\int_{L}\phi\Big(\frac{\langle T^t x, (y_1-z_1)\rangle}{\lambda}\Big)\mathrm{d}y_1\mathrm{d}z_1 \le 1\right\}\\
&= h_{Z_\phi(K,L)}(T^t x) = h_{TZ_\phi(K,L)}(x).
\end{aligned}$$

Therefore, we have $Z_\phi(TK, TL) = T(Z_\phi(K, L))$. □

3 Proof of the affine isoperimetric inequality for Orlicz mean zonoids

In [1], Campi and Gronchi prove that a family of parallel chord movement under the behavior of the L_p-centroid operator is still a shadow system along the same direction. This result leads another proof of the L_p-Busemann-Petty centroid inequality. The similar results for Orlicz centroid operator also hold, which is obtained in [16]. We first prove the following results.

Theorem 3.1. *If $\{K_t : t \in [0,1]\}$ and $\{L_t : t \in [0,1]\}$ are parallel chord movement along the direction v, respectively. Then $Z_\phi(K_t, L_t)$ is a shadow system along the same direction v.*

To prove Theorem 3.1, the following some lemmas will be needed.

Lemma 3.2. *If $\{K_t : t \in [0,1]\}$ and $\{L_t : t \in [0,1]\}$ are parallel chord movement along the unit direction v, respectively. Then the orthogonal projection of $Z_\phi(K_t, L_t)$ onto $v^\perp$ is independent of t.*

Proof. By (1.2) and (2.2), we have

$$\begin{aligned}
&h_{Z_\phi(K_t,L_t)}(x)\\
&= \inf\left\{\lambda > 0 : \frac{1}{V(K_t)V(L_t)}\int_{K_t}\int_{L_t}\phi\Big(\frac{\langle x, (y-z)\rangle}{\lambda}\Big)\mathrm{d}y\mathrm{d}z \le 1\right\}\\
&= \inf\left\{\lambda > 0 : \frac{1}{V(K_0)V(L_0)}\int_{K_0}\int_{L_0}\phi\Big(\frac{\langle x, (y+\beta(y|v^\perp)tv - z - \beta(z|v^\perp)tv)\rangle}{\lambda}\Big)\mathrm{d}y\mathrm{d}z \le 1\right\}\\
&= \inf\left\{\lambda > 0 : \frac{1}{V(K)V(L)}\int_{K}\int_{L}\phi\Big(\frac{\langle x, (y-z)\rangle + (\beta(y|v^\perp) - \beta(z|v^\perp))t\langle x, v\rangle}{\lambda}\Big)\mathrm{d}y\mathrm{d}z \le 1\right\}.
\end{aligned}$$

Then for $x \in v^{\perp}, h_{Z_\phi(K_t,L_t)}(x) = h_{Z_\phi(K,L)}(x)$. □

The following lemma shows that $h_{Z_\phi(K_t,L_t)}(x)$ is a Lipschitz function of t, hence is continuous with respect to t.

Lemma 3.3. *If $K, L \in \mathcal{K}^n$ and $\phi \in \mathcal{C}$, then for $t_1, t_2 \in (0,1]$ and $x \in \mathbb{R}^n \backslash \{o\}$,*

$$|h_{Z_\phi(K_{t_1},L_{t_1})}(x) - h_{Z_\phi(K_{t_2},L_{t_2})}(x)| \le |t_1 - t_2| \; \|(\beta(y|v^{\perp}) - \beta(z|v^{\perp}))\langle x, v\rangle\|_\phi,$$

where $\|\cdot\|_\phi$ is defined for $f : \mathcal{K}^n \times \mathcal{K}^n \to \mathbb{R}$ which is continuous and not constant to 0 as

$$\|f\|_\phi = \inf\left\{\lambda > 0 : \frac{1}{V(K)V(L)}\int_K\int_L \phi\Big(\frac{f(y,z)}{\lambda}\Big)\mathrm{d}y\mathrm{d}z \le 1\right\}.$$

Proof. Let $f, g : \mathcal{K}^n \times \mathcal{K}^n \to \mathbb{R}$ be continuous and not constant to 0. Then the strict convexity of ϕ on $\mathbb{R}$ implies that

$$\|f\|_\phi = \lambda_1 \quad \Leftrightarrow \quad \frac{1}{V(K)V(L)}\int_K\int_L \phi\Big(\frac{f(y,z)}{\lambda_1}\Big)\mathrm{d}y\mathrm{d}z = 1 \tag{3.1}$$

and

$$\|g\|_\phi = \lambda_2 \quad \Leftrightarrow \quad \frac{1}{V(K)V(L)}\int_K\int_L \phi\Big(\frac{g(y,z)}{\lambda_2}\Big)\mathrm{d}y\mathrm{d}z = 1. \tag{3.2}$$

The convexity of the function ϕ shows that

$$\phi\Big(\frac{f(y,z)+g(y,z)}{\lambda_1+\lambda_2}\Big) \le \frac{\lambda_1}{\lambda_1+\lambda_2}\phi\Big(\frac{f(y,z)}{\lambda_1}\Big) + \frac{\lambda_2}{\lambda_1+\lambda_2}\phi\Big(\frac{g(y,z)}{\lambda_1}\Big).$$

Integrating both sides with respect to the Lebesgue measure of K, L and using (3.1), (3.2) give

$$\frac{1}{V(K)V(L)}\int_K\int_L \phi\Big(\frac{f(y,z)+g(y,z)}{\lambda_1+\lambda_2}\Big)\mathrm{d}y\mathrm{d}z \le 1.$$

From the definition of $\|\cdot\|_\phi$ we get

$$\|f+g\|_\phi \le \lambda_1 + \lambda_2 = \|f\|_\phi + \|g\|_\phi.$$

Thus

$$|\;\|f\|_\phi - \|g\|_\phi| \le \|f-g\|_\phi.$$

The facts that ϕ is even and

$$h_{Z_\phi(K_t,L_t)}(x) = \|\langle x, y-z\rangle + (\beta(y|v^{\perp}) - \beta(z|v^{\perp}))t\langle x, v\rangle\|_\phi$$

conclude the proof. □

Lemma 3.4. (see [5]) *If $K, L \in \mathcal{K}^n$, then $h_{Z_\phi(K,L)}$ is the support function of a convex body and the operation Z_ϕ maps $\mathcal{K}^n \times \mathcal{K}^n$ to $\mathcal{K}^n_o$.*

From Lemma 3.4, since $Z_\phi(K_t, L_t)$ is a convex body for every $t \in [0,1]$, it can be represented by

$$Z_\phi(K_t, L_t) = \{x + yv : x \in (Z_\phi(K_0, L_0))|v^{\perp}, f_t(x) \le y \le g_t(x)\}, \tag{3.3}$$

where f_t and $-g_t$ are convex functions defined on $(Z_\phi(K_0, L_0))|v^{\perp}$.

Lemma 3.5. *If $\{K_t : t \in [0,1]\}$ and $\{L_t : t \in [0,1]\}$ are parallel chord movement along the unit direction v, respectively. Then for every $x \in (Z_\phi(K_0, L_0))|v^\perp$,*

$$g_t(x) = \inf_{u \in v^\perp} \{h_{Z_\phi(K_t,L_t)}(u+v) - \langle x, u\rangle\}, \tag{3.4}$$

and

$$f_t(x) = \sup_{u \in v^\perp} \{\langle x, u\rangle - h_{Z_\phi(K_t,L_t)}(u-v)\}. \tag{3.5}$$

Proof. Let $u \in v^\perp$. For $x \in (Z_\phi(K_0, L_0))|v^\perp$ we have

$$x + g_t(x)v \in Z_\phi(K_t, L_t), \qquad x + f_t(x)v \in Z_\phi(K_t, L_t).$$

The definition of the support function shows that

$$\langle x + g_t(x)v, u+v\rangle \le h_{Z_\phi(K_t,L_t)}(u+v),$$

$$\langle x + f_t(x)v, u-v\rangle \le h_{Z_\phi(K_t,L_t)}(u-v).$$

Thus, for all $u \in v^\perp$,

$$\langle x, u\rangle + g_t(x) \le h_{Z_\phi(K_t,L_t)}(u+v),$$
$$\langle x, u\rangle - f_t(x) \le h_{Z_\phi(K_t,L_t)}(u-v).$$

Since $Z_\phi(K_t, L_t)$ has support hyperplanes at the two points $x+g_t(x)v, x+f_t(x)v \in \partial(Z_\phi(K_t, L_t))$, for $x \in \text{relint}((Z_\phi(K_0, L_0)|v^\perp)$, there exist two vectors $u'+v$ and $u''-v$ with $u', u'' \in v^\perp$ such that

$$\langle x + g_t(x)v, u'+v\rangle = h_{Z_\phi(K_t,L_t)}(u'+v),$$
$$\langle x + f_t(x)v, u''-v\rangle = h_{Z_\phi(K_t,L_t)}(u''-v).$$

If $x \notin \text{relint}((Z_\phi(K_0, L_0)|v^\perp)$, it is possible that $g_t(x) = 0, f_t(x) = 0$. Then we cannot find $u', u'' \in v^\perp$ such that

$$\langle x + g_t(x)v, u'+v\rangle = \langle x, u'\rangle = h_{Z_\phi(K_t,L_t)}(u'+v),$$

$$\langle x + f_t(x)v, u''-v\rangle = \langle x, u''\rangle = h_{Z_\phi(K_t,L_t)}(u''-v).$$

The continuity of support functions ensures that we can take the infimum and supremum for all $u \in v^\perp$. Therefore, we get

$$g_t(x) = \inf_{u \in v^\perp} \{h_{Z_\phi(K_t,L_t)}(u+v) - \langle x, u\rangle\}$$

and

$$f_t(x) = \sup_{u \in v^\perp} \{\langle x, u\rangle - h_{Z_\phi(K_t,L_t)}(u-v)\}$$

for every $x \in (Z_\phi(K_0, L_0))|v^\perp$. □

Since $h_{Z_\phi(K_t,L_t)}(x)$ is a Lipschitz function of t, with Lipschitz constant $\|(\beta(y|v^\perp) - \beta(z|v^\perp))\langle x, v\|_\phi$, from Lemma 3.5 we deduce that $g_t(x)$ and $-f_t(x)$ are Lipschitz functions of t too. Therefore $g_t(x)$ and $f_t(x)$ are continuous with respect to t. Additionally, the convexity of $g_t(x)$ and $-f_t(x)$ with respect to t can be stated as follows.

Lemma 3.6. *If* $\{K_t : 0 \le t \le 1\}$ *and* $\{L_t : 0 \le t \le 1\}$ *are parallel chord movement along the unit direction* v*, respectively. Then for every* $x \in (Z_\phi(K_0, L_0))|v^\perp$, $g_t(x)$ *and* $-f_t(x)$ *are convex functions of the parameter* t *in* $[0,1]$.

Proof. We first show that if $u_1, u_2 \in v^\perp$, then

$$h_{Z_\phi\left(K_{\frac{t_1+t_2}{2}}, L_{\frac{t_1+t_2}{2}}\right)}(u_1 + u_2 + 2v) \le h_{Z_\phi(K_{t_1}, L_{t_1})}(u_1 + v) + h_{Z_\phi(K_{t_2}, L_{t_2})}(u_2 + v). \tag{3.6}$$

Indeed, let $h_{Z_\phi(K_{t_1}, L_{t_1})}(u_1 + v) = \lambda_1$, $h_{Z_\phi(K_{t_2}, L_{t_2})}(u_2 + v) = \lambda_2$, and write $\beta_v(y,z) = \beta(y|v^\perp) - \beta(z|v^\perp)$. The convexity of ϕ gives that

$$\begin{aligned}
&\phi\left(\frac{\langle u_1 + u_2 + 2v, y - z\rangle + \beta_v(y,z)\frac{t_1+t_2}{2}\langle u_1 + u_2 + 2v, v\rangle}{\lambda_1 + \lambda_2}\right) \\
&= \phi\left(\frac{\langle u_1 + v, y - z\rangle + \beta_v(y,z)t_1 + \langle u_2 + v, y - z\rangle + \beta_v(y,z)t_2}{\lambda_1 + \lambda_2}\right) \\
&= \phi\left(\frac{\langle u_1 + v, y - z\rangle + \beta_v(y,z)t_1\langle u_1 + v, v\rangle}{\lambda_1 + \lambda_2}\right. \\
&\quad \left. + \frac{\langle u_2 + v, y - z\rangle + \beta_v(y,z)t_2\langle u_2 + v, v\rangle}{\lambda_1 + \lambda_2}\right) \\
&\le \frac{\lambda_1}{\lambda_1 + \lambda_2}\phi\left(\frac{\langle u_1 + v, y - z\rangle + \beta_v(y,z)t_1\langle u_1 + v, v\rangle}{\lambda_1}\right) \\
&\quad + \frac{\lambda_2}{\lambda_1 + \lambda_2}\phi\left(\frac{\langle u_2 + v, y - z\rangle + \beta_v(y,z)t_2\langle u_2 + v, v\rangle}{\lambda_2}\right).
\end{aligned} \tag{3.7}$$

Integrating both sides and using (2.4), we obtain (3.6).

It follows from Lemma 3.5 and (3.6) that

$$\begin{aligned}
2g_{\frac{t_1+t_2}{2}}(x) &= \inf_{u \in v^\perp}\left\{h_{Z_\phi(K_{\frac{t_1+t_2}{2}}, L_{\frac{t_1+t_2}{2}})}(2(u+v)) - \langle x, 2u\rangle\right\} \\
&= \inf_{u_1, u_2 \in v^\perp}\left\{h_{Z_\phi(K_{\frac{t_1+t_2}{2}}, L_{\frac{t_1+t_2}{2}})}(u_1 + u_2 + 2v) - \langle x, u_1 + u_2\rangle\right\} \\
&\le \inf_{u_1, u_2 \in v^\perp}\left\{h_{Z_\phi(K_{t_1}, L_{t_1})}(u_1 + v) + h_{Z_\phi(K_{t_2}, L_{t_2})}(u_1 + v) - \langle x, u_1 + u_2\rangle\right\} \\
&= \inf_{u_1 \in v^\perp}\left\{h_{Z_\phi(K_{t_1}, L_{t_1})}(u_1 + v) - \langle x, u_1\rangle\right\} \\
&\quad + \inf_{u_2 \in v^\perp}\left\{h_{Z_\phi(K_{t_2}, L_{t_2})}(u_2 + v) - \langle x, u_2\rangle\right\} \\
&= g_{t_1}(x) + g_{t_2}(x).
\end{aligned}$$

The convexity of the function $-f_t$ of t can be proved in the same way. □

Lemma 3.7. *If* $\{K_t : t \in [0,1]\}$ *and* $\{L_t : t \in [0,1]\}$ *are parallel chord movement along the unit direction* v*, respectively. Then for every* $x \in (Z_\phi(K_0, L_0))|v^\perp$ *and* $t_1, t_2, \theta \in [0,1]$,

$$f_{\theta t_1 + (1-\theta)t_2}(x) \le \theta g_{t_1}(x) + (1-\theta) f_{t_2}(x) \le g_{\theta t_1 + (1-\theta)t_2}(x).$$

Proof. Let $u_1, u_2 \in v^\perp$ and

$$h_{Z_\phi(K_{t_1},L_{t_1})}(-\theta u_1+\theta v)=\lambda_1,\quad h_{Z_\phi(K_{\theta t_1+(1-\theta)t_2},L_{\theta t_1+(1-\theta)t_2})}(u_2-v)=\lambda_2.$$

Then we have

$$\begin{aligned}
&\phi\Big(\frac{\langle u_2-\theta u_1-(1-\theta)v,y-z\rangle+\beta_v(y,z)t_2\langle u_2-\theta u_1-(1-\theta)v,v\rangle}{\lambda_1+\lambda_2}\Big)\\
&\quad=\phi\Big(\frac{\langle u_2-v,y-z\rangle+\langle-\theta u_1+\theta v,y-z\rangle-\beta_v(y,z)((1-\theta)t_2+\theta t_1-\theta t_1)}{\lambda_1+\lambda_2}\Big)\\
&\quad\le\frac{\lambda_2}{\lambda_1+\lambda_2}\phi\Big(\frac{\langle u_2-v,y-z\rangle+\beta_v(y,z)((1-\theta)t_2+\theta t_1)\langle u_2-v,v\rangle}{\lambda_2}\Big)\\
&\qquad+\frac{\lambda_1}{\lambda_1+\lambda_2}\phi\Big(\frac{\langle-\theta u_1+\theta v,y-z\rangle+\beta_v(y,z)t_1\langle-\theta u_1+\theta v,v\rangle}{\lambda_1}\Big).
\end{aligned}$$

Integrating both sides and using (2.4) give

$$\begin{aligned}
&h_{Z_\phi(K_{t_2},L_{t_2})}(u_2-\theta u_1-(1-\theta)v)\\
&\quad\le h_{Z_\phi(K_{t_1},L_{t_1})}(-\theta u_1+\theta v)+h_{Z_\phi(K_{\theta t_1+(1-\theta)t_2},K_{\theta t_1+(1-\theta)t_2})}(u_2-v).
\end{aligned}\tag{3.8}$$

Thus, from (3.8) and Lemma 3.5, we get

$$\begin{aligned}
&(1-\theta)f_{t_2}(x)\\
=\;&\sup_{u\in v^\perp}\{\langle x,(1-\theta)u\rangle-h_{Z_\phi(K_{t_2},L_{t_2})}((1-\theta)(u-v))\}\\
=\;&\sup_{-u_1,u_2\in v^\perp}\{\langle x,u_2-\theta u_1\rangle-h_{Z_\phi(K_{t_2},L_{t_2})}(u_2-\theta u_1-(1-\theta)v)\}\\
\ge\;&\sup_{-u_1,u_2\in v^\perp}\{\langle x,u_2-\theta u_1\rangle-h_{Z_\phi(K_{t_1},L_{t_1})}(-\theta u_1+\theta v)\\
&\quad-h_{Z_\phi(K_{\theta t_1+(1-\theta)t_2},L_{\theta t_1+(1-\theta)t_2})}(u_2-v)\}\\
=\;&\sup_{-u_1\in v^\perp}\{\langle x,-\theta u_1\rangle-h_{Z_\phi(K_{t_1},L_{t_1})}(-\theta u_1+\theta v)\}\\
&\quad+\sup_{u_2\in v^\perp}\{\langle x,u_2\rangle-h_{Z_\phi(K_{\theta t_1+(1-\theta)t_2},L_{\theta t_1+(1-\theta)t_2})}(u_2-v)\}\\
=\;&-\theta g_{t_1}(x)+f_{\theta t_1+(1-\theta)t_2}(x).
\end{aligned}$$

This gives the first inequality. The second inequality follows by interchanging t_1 with t_2 and x with $-x$. □

In order to prove Theorem 3.1 we shall require the following crucial lemma, which was proved by Campi and Gronchi [2].

Lemma 3.8. *If $\{H_t : t\in[0,1]\}$ be a one-parameter family of convex bodies such that $H_t|v^\perp$ is independent of t. Assume that the bodies H_t are defined by*

$$H_t=\{x+yv:\ x\in H_t|v^\perp, y\in\mathbb{R}, f_t(x)\le y\le g_t(x)\},\quad t\in[0,1],$$

for suitable functions g_t and f_t. Then $\{H_t : t\in[0,1]\}$ is a shadow system along the direction v if and only if for every $x\in H_0|v^\perp$,

(1) $g_t(x)$ and $-f_t(x)$ are convex functions of the parameter t in $[0,1]$,

(2) $f_{\lambda t_1+(1-\lambda)t_2}(x)\le\lambda g_{t_1}(x)+(1-\lambda)f_{t_2}(x)\le g_{\lambda t_1+(1-\lambda)t_2}(x)$, for every $t_1,t_2,\lambda\in[0,1]$.

Proof of Theorem 3.1. Let $\{K_t : t \in [0,1]\}$ and $\{L_t : t \in [0,1]\}$ are parallel chord movement along the unit direction v, respectively. By Lemma 3.2 we obtain that the orthogonal projection of $Z_\phi(K_t, L_t)$ onto $v^\perp$ is independent of t. Then from (3.3) and Lemma 3.8 it is sufficient to show that the family $Z_\phi(K_t, L_t)$ satisfies conditions (1) and (2) of Lemma 3.8. In fact, Lemma 3.6 and Lemma 3.7 demonstrate these two conditions for $Z_\phi(K_t, L_t)$. Therefore, we deduce that $Z_\phi(K_t, L_t)$ is a shadow system along the direction v. □

Theorem 3.9. *If $\{K_t : t \in [0,1]\}$ and $\{L_t : t \in [0,1]\}$ are parallel chord movement with speed function β, respectively. Then the volume of $Z_\phi(K_t, L_t)$ is a strictly convex function of t unless β is linear function defined on $v^\perp$, that $\beta(x) = \langle x, u\rangle$.*

Proof. By Fubini's theorem it is easy to see that

$$V(Z_\phi(K_t, L_t)) = \int_{(Z_\phi(K_0,L_0))|v^\perp} (g_t(x) - f_t(x))\mathrm{d}x. \tag{3.9}$$

That the volume of $Z_\phi(K_t, L_t)$ is a convex function of t therefore follows from the convexity of $g_t(x)$ and $-f_t(x)$ with respect to t.

Suppose that

$$V\Big(Z_\phi(K_{\frac{t_1+t_2}{2}}, L_{\frac{t_1+t_2}{2}})\Big) = \frac{1}{2}V(Z_\phi(K_{t_1}, L_{t_1})) + \frac{1}{2}V(Z_\phi(K_{t_2}, L_{t_2}))$$

for some $t_1, t_2 \in [0,1]$. From (3.9) and the continuity of g_t, f_t with respect to x, it follows that

$$g_{\frac{t_1+t_2}{2}}(x) - f_{\frac{t_1+t_2}{2}}(x) = \frac{1}{2}\big(g_{t_1}(x) + g_{t_2}(x)\big) - \frac{1}{2}\big(f_{t_1}(x) + f_{t_2}(x)\big) \tag{3.10}$$

for almost every $x \in (Z_\phi(K_0, L_0))|v^\perp$. Let $x \in \operatorname{relint}\big((Z_\phi(K_0, L_0))|v^\perp\big)$. Then there exist $u_1, u_2, u_3, u_4 \in v^\perp$ such that

$$\begin{aligned}
&\frac{1}{2}\big(g_{t_1}(x) + g_{t_2}(x)\big) - \frac{1}{2}\big(f_{t_1}(x) + f_{t_2}(x)\big)\\
&= \frac{1}{2}\Big(h_{Z_\phi(K_{t_1},L_{t_1})}(u_1 + v) + h_{Z_\phi(K_{t_2},L_{t_2})}(u_2 + v)\\
&\quad + h_{Z_\phi(K_{t_1},L_{t_1})}(u_3 - v) + h_{Z_\phi(K_{t_2},L_{t_2})}(u_4 - v)\\
&\quad - \langle x, u_1\rangle - \langle x, u_2\rangle - \langle x, u_3\rangle - \langle x, u_4\rangle\Big).
\end{aligned}$$

By (3.6) and Lemma 3.5 we get

$$\begin{aligned}
&\frac{1}{2}(g_{t_1}(x) + g_{t_2}(x)) - \frac{1}{2}(f_{t_1}(x) + f_{t_2}(x))\\
&\geq h_{Z_\phi\left(K_{\frac{t_1+t_2}{2}},L_{\frac{t_1+t_2}{2}}\right)}\Big(\frac{u_1+u_2}{2} + v\Big) - \Big\langle x, \frac{u_1+u_2}{2}\Big\rangle\\
&\quad + h_{Z_\phi\left(K_{\frac{t_1+t_2}{2}},L_{\frac{t_1+t_2}{2}}\right)}\Big(\frac{u_3+u_4}{2} - v\Big) - \Big\langle x, \frac{u_3+u_4}{2}\Big\rangle\\
&\geq g_{\frac{t_1+t_2}{2}}(x) - f_{\frac{t_1+t_2}{2}}(x).
\end{aligned} \tag{3.11}$$

The equality of (3.10) forces equality in (3.11) and equality in (3.7). Since ϕ is strictly convex, we have

$$\frac{\langle u_1 + v, y - z\rangle + \beta_v(y,z)t_1}{\lambda_1} = \frac{\langle u_2 + v, y - z\rangle + \beta_v(y,z)t_2}{\lambda_2} \tag{3.12}$$

for every $y \in L_0$ and $z \in K_0$, owing to the continuity of β.

Setting $y = y' + sv, y' \in L_0|v^\perp$ and $z = z' - sv, z' \in K_0|v^\perp$, in (3.12) and differentiating with respect to the parameter s, it turns out that $\lambda_1/\lambda_2 = 1$, that is,

$$\langle u_1 + v, y - z\rangle + \beta_v(y,z)t_1 = \langle u_2 + v, y - z\rangle + \beta_v(y,z)t_2.$$

So we conclude that $\beta(x) = \langle x, u\rangle$ for some vector u. This completes the proof. □

The following lemmas will be needed.

Lemma 3.10. (Shephard [21]) *The volume of a shadow system is a convex function of the parameter t.*

Lemma 3.11. (see [5]) *Let $\phi \in \mathcal{C}$, then the operator $Z_\phi : \mathcal{K}^n \times \mathcal{K}^n \to \mathcal{K}^n_o$ is continuous in the Hausdorff metric.*

In order to give a new proof of (1.3), we will an affine isoperimetric inequality for for Orlicz mean zonoid body can be redescribed as follows:

Theorem 3.12. (The affine isoperimetric inequality for Orlicz mean zonoids, see [5]) *If $\phi \in \mathcal{C}$ and $K, L \in \mathcal{K}^n_o$, then*

$$V(Z_\phi(K,L)) \geq V(Z_\phi(B_K, B_L)),$$

with equality if K and L are dilated ellipsoids having the same midpoints.

Proof. Theorem 3.1 and Lemma 3.10 imply that the volume of $Z_\phi(K_t, L_t)$ is a convex function of t. From Lemma 2.3 we get that $Z_\phi(K^v, L^v) = (Z_\phi(K,L))^v$. Thus

$$\begin{aligned} V(Z_\phi(S_vK, S_vL)) &= V(Z_\phi(K_{1/2}, L_{1/2})) \\ &\leq \frac{1}{2}V(Z_\phi(K_0, L_0)) + \frac{1}{2}V(Z_\phi(K_1, L_1)) \\ &= V(Z_\phi(K, L), \end{aligned}$$

that is, the volume of the Orlicz mean zonoid body is not increased after a Steiner symmetrization. The continuity of the Orlicz mean zonoid operator implies the continuity of the volume $V(Z_\phi(K,L))$ in the Hausdorff metric. It follows that the volume attains its minimum value when K and L are ball.

If the speed function β of the parallel chord movement is linear, then it is easy to see that K_t and L_t are linear image of K and L, for every t in the range of the movement, respectively. It is well known, see [19], that if K and L are not origin symmetric ellipsoids respectively, then there exists a direction v such that for the Steiner symmetrization S_vK of K and the Steiner symmetral S_vL of L, it follows that

$$S_vK \neq TK, \quad S_vL \neq TL$$

for all $T \in \mathrm{GL}(n)$. Therefore, $V(Z_\phi(K,L))$ is minimized if and only if K and L are ellipsoids centered at the origin. The affine isoperimetric inequality for Orlicz mean zonoids is established. □

As an extension of the definition (1.2), we introduce the definition of the Orlicz mean zonoid body for star bodies as follows.

Let $K, L \subset \mathbb{R}^n$ be star bodies with respect to the origin and $\phi \in \mathcal{C}$. The Orlicz mean zonoid body $Z_\phi(K, L)$ of K and L as the convex body whose support function at $x \in \mathbb{R}^n$ is defined by

$$h_{Z_\phi(K,L)}(x) = \inf\left\{\lambda > 0 : \frac{1}{V(K)V(L)} \int_K \int_L \phi\left(\frac{\langle x, (y-z)\rangle}{\lambda}\right) \mathrm{d}y\mathrm{d}z \le 1\right\}. \tag{3.13}$$

By definition (3.13), we posted the following open problem:

Conjecture 3.13. *If $\phi \in \mathcal{C}$ and K, L are star bodies with respect to the origin in $\mathbb{R}^n$, then*

$$V(Z_\phi(K, L)) \ge V(Z_\phi(B_K, B_L)),$$

with equality if K and L are dilated ellipsoids having the same midpoints.

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